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# On $L^q$ Norm Inequalities of Polynomial

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***Abstract:***

***If $f(z)=\sum_{v=0}^{n} c_v z^v$ is a polynomial of degree $n$ which does not vanish in $|z|<t$, $t\ge 1$, then for $0<r\le\rho\le t$, Dewan and Mir [ Int. J. Math. Math. Scs., 16(2005), 2641-2645] proved***

$$\max_{|z|=\rho}\left|f'(z)\right| \le n\frac{(\rho+t)^{n-1}}{(t+r)^n}\max_{|z|=r}\left|f(z)\right|$$

$$\times\left\{1-\frac{t(t-\rho)\left(n|c_0|-t|c_1|\right)n}{\left(t^2+\rho^2\right)n|c_0|+2t^2\rho|c_1|}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}.$$

***In this paper, we prove an interesting improved $L^q$ norm inequality with the value of $t$ extending from $t\ge 1$ to $t>0$ of the above inequality. Our result also gives some interesting known results as corollaries.***

## 1. INTRODUCTION

Let $f(z)=\sum_{v=0}^{n} c_v z^v$ be a polynomial of degree $n$ and $f'(z)$ be its derivative. We define

$$\|f\|_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}, 0<q<\infty \qquad (1.1)$$

If we let $q\to\infty$ in the above equality and make use of of the well-known fact from analysis [17] that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}=\max_{|z|=1}\left|f(z)\right|,$$

we can suitably denote

$$\|f\|_\infty=\max_{|z|=1}\left|f(z)\right|.$$

Similarly, one can define $\|f\|_0=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(e^{i\theta}\right)\right|d\theta\right\}$ and show that $\lim_{q\to 0+}\|f\|_q=\|f\|_0$. It would be of further interest that by taking limits as $q\to 0+$, the stated result holding for $q>0$, holds fo $q=0$ as well.

For $r>0$, we denote by $M(f,r)=\max_{|z|=r}\left|f(z)\right|$ and accordingly $\|f\|_\infty=\max_{|z|=1}\left|f(z)\right|=M(f,1)$.

A famous result due to Bernstein [14 or also see 19] states that if $f(z)$ is a polynomial of degree $n$, then

$$\|f'\|_\infty\le n\|f\|_\infty. \qquad (1.2)$$

Inequality (1.2) can be obtained by letting $q\to\infty$ in the inequality

$$\|f'\|_q\le n\|f\|_q. \qquad (1.3)$$

Inequality (1.3) for $q\ge 1$ is due to Zygmund [20]. Arestov [1] proved that (1.3) remains valid for $0<q<1$ as well.

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then inequalities (1.2) and (1.3) can be respectively improved by

$$\|f'\|_\infty\le\frac{n}{2}\|f\|_\infty. \qquad (1.4)$$

and

$$\|f'\|_q\le\frac{n}{\|1+z\|_q}\|f\|_q,\ q>0. \qquad (1.5)$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [12], whereas, inequality (1.5) was proved by de-Bruijn [4] for $q \ge 1$. Rahman and Schmeisser [16] showed that (1.5) remains true for $0 < q < 1$.

As a generalization of (1.4), Malik [13] proved that if $f(z)$ does not vanish in $|z| < t$, $t \ge 1$, then

$$\|f'\|_\infty \le \frac{n}{1+t}\|f\|_\infty . \qquad (1.6)$$

Under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [8] extended inequality (1.6) to $L^q$ norm by showing that

$$\|f'\|_q \le \frac{n}{\|t+z\|_q}\|f\|_q , \; q \ge 1 . \qquad (1.7)$$

It was shown by Gardner and Weems [7] and independently by Rather [17] that (1.7) also holds for $0 < q < 1$.

Further, as a generalization of (1.6), Bidkham and Dewan [3] proved that if $f(z)$ is a polynomial of degree $n$ having no zero in $|z| < t$, $t \ge 1$, then for $1 \le r \le t$,

$$\|f'(rz)\|_\infty \le n\frac{(r+t)^{n-1}}{(1+t)^n}\|f\|_\infty \text{ for } 1 \le r \le t . \qquad (1.8)$$

For the same class of polynomials $f(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$, by involving certain coefficients, Dewan and Mir [5] improved as well as generalized inequality (1.8) by proving

$$\|f'(\rho z)\|_\infty \le n\frac{(\rho+t)^{n-1}}{(r+t)^n} \times\left\{1-\frac{t(t-\rho)(n|c_0|-|c_1|t)n}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\|f(rz)\|_\infty \qquad (1.9)$$

for $0 \le r \le \rho \le t$.

In this paper, we prove an improved inequality in $L^q$ norm for extended value of $t > 0$, which not only reduces to $L^q$ version of inequality (1.9) as a particular case, but also gives some interesting known results as corollaries. More precisely, we prove

**Theorem.** *If* $f(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ *is a polynomial of degree* $n$ *having no zero in* $|z| < t$, $t > 0$, *then for* $0 \le r \le \rho \le t$ and any $q > 0$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f'(\rho e^{i\theta})\right|^q d\theta\right\}^{\frac{1}{q}} \le nT_q\left\{\frac{1}{2\pi}\int_o^{2\pi}\left[\left|f(re^{i\theta})\right|+M(f,r)\right.\right.$$
$$\left.\left.\times\left\{\left\{\frac{(\rho^2+t^2)n|c_0|+2t^2|c_1|\rho}{(r^2+t^2)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right\}\right]^q d\theta\right\}^{\frac{1}{q}}, \qquad (1.10)$$

where

$$T_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}} .$$

**Remark 1.1.** If we let $q \to \infty$ on both sides of inequality (1.10) of our theorem, as mentioned earlier, we obtain an improved counterpart of inequality (1.9) as given below.

**Corollary 1.1.** *If* $f(z) = \sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ is a polynomial of degree $n$ having no zero in $|z| < t$, $t > 0$, then for $0 \le r \le \rho \le t$.

$$\|f'(\rho z)\|_\infty \le \frac{n}{\rho+t}\left\{\frac{(\rho^2+t^2)n|c_0|+2t^2|c_1|\rho}{(r^2+t^2)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}\|f(rz)\|_\infty . \qquad (1.11)$$

**Remark 1.2.** If $0 \le r < \rho \le t$, we have

$$1-\left(\frac{t+r}{t+\rho}\right)^n = \frac{(\rho-r)}{(\rho+t)\left\{1-\left(\frac{r+t}{\rho+t}\right)\right\}}\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}$$
$$=\left(\frac{\rho-r}{t+\rho}\right)\left\{\left(\frac{t+r}{t+\rho}\right)^{n-1}+\left(\frac{t+r}{t+\rho}\right)^{n-2}+......+\left(\frac{t+r}{t+\rho}\right)+1\right\}$$
$$\ge\left(\frac{\rho-r}{t+\rho}\right)n\left(\frac{t+r}{t+\rho}\right)^{n-1} . \qquad (1.12)$$

Also, for $r=\rho$, inequality (1.12) holds trivially and hence inequality (1.12) is true for $0\le r\le\rho\le t$. By Lemma 2.10, we have

$$\left\{\frac{\left(\rho^2+t^2\right)n|c_0|+2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}\le\frac{\left(n|c_0|\rho+|c_1|t^2\right)(\rho+t)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\times\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}+1$$

$$\le\left(\frac{\rho+t}{r+t}\right)^n\left[1-\left\{\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}\times\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\right]$$

(by Lemma 2.9)

$$\le\frac{(\rho+t)^n}{(r+t)^n}\left\{1-\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)n}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\times\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}\text{ [ by inequality(1.12)]}$$

(1.13)

It is interesting that by using this inequality in inequality (1.10), we obtain the direct $L^q$ analogue of (1.9) due to Dewan and Mir [5] with extended value of the radius $t$ of the zero free open disc from $t\ge1$ to $t>0$.

**Corollary 1.2.** If $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ is a polynomial of degree $n$ having no zero in $|z|<t$, $t>0$, then for $0\le r\le\rho\le t$ and any $q>0$,

$$\left\|f'(\rho z)\right\|_q\le nT_q\left\|\left|f(rz)\right|+M(f,r)\left\{\frac{(\rho+t)^n}{(r+t)^n}\times\left\{1-\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)n}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left(\frac{\rho-r}{t+\rho}\right)\left(\frac{t+r}{t+\rho}\right)^{n-1}\right\}-1\right\}\right\|_q,$$

(1.14)

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

**Remark 1.3.** Taking limit as $q\to\infty$ on both sides of (1.14), we obtain inequality (1.9).

**Remark 1.4.** If we use the fact that $\left|f\left(re^{i\theta}\right)\right|\le M(f,r)=\left\|f(rz)\right\|_\infty$ for each $\theta\in[0,\,2\pi)$, we obtain another improved version of inequality (1.9) in $L^q$ norm deduced from our theorem.

**Corollary 1.3.** *If* $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ is a polynomial of degree $n$ having no zero in $|z|<t$, $t>0$, then for $0\le r\le\rho\le t$ and any $q>0$,

$$\left\|f'(\rho z)\right\|_q\le nT_q\left\{\frac{\left(\rho^2+t^2\right)n|c_0|+2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}\left\|f(rz)\right\|_\infty. \tag{1.15}$$

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|t+\rho e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

**Remark 1.5.** Putting $r=1$ and replacing $\rho$ by $r$ in corollary 1.3, we have an improvement of (1.8). Further, putting $1=r=\rho$, corollary 1.2 reduce to inequality (1.7). Also assigning $1=r=\rho=k$, both the theorem and corollary 1.2 reduce to the well-known de- Bruijn inequality (1.5).

## 2. LEMMA

The following lemmas are needed for the proof of the theorem.

**Lemma 2.1.** *If* $f(z)=\sum_{\upsilon=1}^{n}c_\upsilon z^\upsilon$ *is a polynomial of degree* $n$ *having no zero in* $|z|<t$, $t\ge1$, *then*

$$\max_{|z|=1}\left|f'(z)\right|\le\frac{n}{1+t}\max_{|z|=1}\left|f(z)\right|. \tag{2.1}$$

This result is due to Malik [13].

**Lemma 2.2.** *If* $f(z)=\sum_{\upsilon=1}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then for* $|z|=1$,

$$t|f(z)|\leq |g'(z)|. \qquad (2.2)$$

where $g(z)=z^n\overline{f\left(\frac{1}{\bar{z}}\right)}$.

Malik [13, Lemma 3] proved this lemma.

**Lemma 2.3.** *If* $f(z)=\sum_{\upsilon=1}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then*

$$\max_{|z|=1}|f'(z)|\leq n\frac{n|c_0|+t^2|c_1|}{(1+t^2)n|c_0|+2t^2|c_1|}\max_{|z|=1}|f(z)|. \qquad (2.3)$$

This result was proved by Govil at. el. [9].

**Lemma 2.4.** *If* $f(z)=c_0+\sum_{\upsilon=\mu}^{n} c_{\upsilon} z^{\upsilon}$, $1\leq \mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<t$, $t\geq 1$, *then*

$$\frac{\mu}{n}\left|\frac{c_\mu}{c_0}\right|t^{\mu}\leq 1. \qquad (2.4)$$

Lemma 2.3 is due to Qazi [15, Remark 1].

**Lemma 2.5.** *If* $f(z)=\sum_{\upsilon=0}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then the function*

$$p(x)=\frac{\left(n|c_0|x+|c_1|t^2\right)(t+x)}{\left(x^2+t^2\right)n|c_0|+2|c_1|t^2x}. \qquad (2.5)$$

*is a non-decreasing function of* $x$ *in* $(0,t]$.

**Proof of Lemma 2.5.** We prove this by derivative test. Now, we have

$$p'(x)=\frac{\left(n|c_0|-|c_1|t\right)}{\left\{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x\right\}^2}$$
$$\times\left\{(t-x)t\left(n|c_0|x+|c_1|t^2\right)+\left(n|c_0|+|c_1|t\right)\left(t^2x+t^3\right)\right\},$$

which is non-negative, since by Lemma 2.4, for $\mu=1$, $\left(n|c_0|-|c_1|t\right)\geq 0$, and the fact that $x\leq t$.

**Lemma 2.6.** If $f(z)=\sum_{\upsilon=0}^{n} c_{\upsilon} z^{\upsilon}$ is a polynomial of degree n having no zero in $|z|<t$, $t>0$ then for $0\leq r\leq \rho\leq t$,

$$\max_{|z|=\rho}|f(z)|\leq\left(\frac{\rho+t}{r+t}\right)^n\max_{|z|=r}|f(z)|. \qquad (2.6)$$

This lemma is due to Jain [10].

**Lemma 2.7.** If $f(z)=\sum_{\upsilon=0}^{n} c_{\upsilon} z^{\upsilon}$ is a polynomial of degree n having no zero in $|z|<t$, $t>0$ then for $0\leq r\leq \rho\leq t$,

$$\max_{|z|=\rho}|f(z)|\leq\left(\frac{\rho^2+t^2+2t\rho|\delta|}{r^2+t^2+2tr|\delta|}\right)^{\frac{n}{2}}\max_{|z|=r}|f(z)|. \qquad (2.7)$$

where

$$\delta=\frac{tc_1}{nc_0}\ \textit{and}\ |\delta|\leq 1.$$

This lemma was proved by Jain [11, see Remark 1]. Further, he mentioned that the function $\frac{\rho^2+t^2+2t\rho|\delta|}{r^2+t^2+2tr|\delta|}$ is a decreasing function of $|\delta|$ in $[0,1]$ and as $|\delta|\leq 1$, it is concluded that

$$\left(\frac{\rho^2+t^2+2t\rho|\delta|}{r^2+t^2+2tr|\delta|}\right)^{\frac{n}{2}}\leq\left(\frac{\rho+t}{r+t}\right)^n, \qquad (2.8)$$

which clearly implies that the bound of Lemma 2.7 improves over that of Lemma 2.6.

**Lemma 2.8**. *If* $f(z)=\sum_{\upsilon=0}^{n} c_{\upsilon} z^{\upsilon}$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then for* $0\leq r\leq\rho\leq t$,

$$\left|f\left(\rho e^{i\theta}\right)\right|\leq\left|f\left(re^{i\theta}\right)\right|+M(f,r)$$
$$\times\left[\left\{\frac{\left(\rho^2+t^2\right)n|c_0|+2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right]. \qquad (2.9)$$

**Proof of Lemma 2.8.** Since $f(z)$ does no vanish in $|z|<t$, $t>0$, the polynomial $F(z)=f(xz)$ where $0<x\le t$ has no zero in $|z|<\frac{t}{x}$, where $\frac{t}{x}\ge 1$. Hence applying Lemma 2.3 to the polynomial $F(z)$, we get

$$\max_{|z|=1}|F'(z)|\le n\left[\frac{n|c_0|+|c_1x|\left(\frac{t}{x}\right)^2}{\left\{1+\left(\frac{t}{x}\right)^2\right\}n|c_0|+2|c_1x|\left(\frac{t}{x}\right)^2}\right]\max_{|z|=1}|F(z)|,$$

which implies

$$\max_{|z|=x}|f'(z)|\le n\left\{\frac{n|c_0|x+|c_1|t^2}{(x^2+t^2)n|c_0|+2t^2|c_1|x}\right\}\max_{|z|=x}|f(z)|. \tag{2.10}$$

Now, for $0<r\le\rho\le t$ and $0\le\theta<2\pi$, we have on using (2.10)

$$\left|f(\rho e^{i\theta})-f(re^{i\theta})\right|\le\int_r^\rho\left|f'(xe^{i\theta})\right|dx$$

$$\le\int_r^\rho n\left\{\frac{n|c_0|x+|c_1|t^2}{(x^2+t^2)n|c_0|+2t^2|c_1|x}\right\}\max_{|z|=x}|f(z)|dx,$$

which on applying Lemma 2.7 gives

$$\left|f(\rho e^{i\theta})-f(re^{i\theta})\right|\le\int_r^\rho n\left\{\frac{n|c_0|x+|c_1|t^2}{(x^2+t^2)n|c_0|+2t^2|c_1|x}\right\}$$

$$\times\left(\frac{x^2+t^2+2t\,x|\delta|}{r^2+t^2+2t\,r|\delta|}\right)^{\frac{n}{2}}M(f,r)dx \tag{2.11}$$

$$=nM(f,r)\int_r^\rho\left\{\frac{n|c_0|x+|c_1|t^2}{(x^2+t^2)n|c_0|+2t^2|c_1|x}\right\}$$

$$\times\left(\frac{x^2+t^2+2t\,x|\delta|}{r^2+t^2+2t\,r|\delta|}\right)^{\frac{n}{2}}dx \tag{2.12}$$

Substituting the value of $\delta=\frac{tc_1}{nc_0}$, (2.12) is equivalent to

$$\left|f(\rho e^{i\theta})-f(re^{i\theta})\right|\le nM(f,r)$$

$$\times\int_r^\rho\left\{\frac{n|c_0|x+|c_1|t^2}{(x^2+t^2)n|c_0|+2t^2|c_1|x}\right\}\left\{\frac{(x^2+t^2)n|c_0|+2t^2|c_1|x}{(r^2+t^2)n|c_0|+2t^2r|c_1|}\right\}^{\frac{n}{2}}dx$$

$$=\frac{nM(f,r)}{\left((r^2+t^2)n|c_0|+2t^2r|c_1|\right)^{\frac{n}{2}}}$$

$$\times\int_r^\rho\left\{(x^2+t^2)n|c_0|+2t^2|c_1|x\right\}^{\frac{n}{2}-1}\left(n|c_0|x+|c_1|t^2\right)dx$$

$$=M(f,r)\left[\left\{\frac{(\rho^2+t^2)n|c_0|+2t^2|c_1|\rho}{(r^2+t^2)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right],$$

from which it is implied by triangle inequality that

$$\left|f(\rho e^{i\theta})\right|\le\left|f(re^{i\theta})\right|+M(f,r)$$

$$\times\left[\left\{\frac{(\rho^2+t^2)n|c_0|+2t^2|c_1|\rho}{(r^2+t^2)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right],$$

which completes the proof of Lemma 2.8.

**Lemma 2.9**. *If* $f(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then for* $0<r\le\rho\le t$,

$$\frac{(\rho+t)(n|c_0|\rho+|c_1|t^2)}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}=\left(\frac{\rho+t}{r+t}\right)^n$$

$$\times\left[1-\left\{\frac{t(t-\rho)(n|c_0|-|c_1|t)}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\right\}\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\right]-1.$$

**Proof of Lemma 2.9.** We have

$$\frac{(\rho+t)(n|c_0|\rho+|c_1|t^2)}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}=\left(\frac{\rho+t}{r+t}\right)^n$$

$$\times\left\{1-\left(\frac{t+r}{t+\rho}\right)^n\right\}\left\{\frac{(\rho+t)(n|c_0|\rho+|c_1|t^2)}{(\rho^2+t^2)n|c_0|+2\rho|c_1|t^2}\right\}. \tag{2.13}$$

Now

$$\left\{\frac{(\rho+t)\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}=1-\left\{\frac{t(t-\rho)\left(n|c_0|-|c_1|t\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\right\}.$$

(2.14)

Using (2.14) in the right hand side (2.13), we get the required result.

**Lemma 2.10**. *If* $f(z)=\sum_{\upsilon=0}^{n} c_\upsilon z^\upsilon$ *is a polynomial of degree n having no zero in* $|z|<t$, $t>0$, *then for* $0<r\le\rho\le t$,

$$\left[\left\{\frac{\left(\rho^2+t^2\right)n|c_0|+2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right],$$

$$\le\frac{\left(n|c_0|\rho+|c_1|t^2\right)(\rho+t)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}$$

*where* $\delta$ *is as defined in Lemma* 2.7.

**Proof of Lemma 2.10.**

Consider the integral

$$I=\int_r^\rho n\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x}\right\}\left(\frac{x^2+t^2+2t\,x|\delta|}{r^2+t^2+2t\,r|\delta|}\right)^{\frac{n}{2}}dx$$

(2.15)

By inequality (2.8) of Lemma 2.7, we have

$$\left(\frac{x^2+t^2+2t\,x|\delta|}{r^2+t^2+2t\,r|\delta|}\right)^{\frac{n}{2}}\le\left(\frac{t+x}{t+r}\right)^n,$$

using this inequality in (2.15), we obtain

$$I\le\int_r^\rho n\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|x}\right\}\left(\frac{t+x}{t+r}\right)^n dx$$

$$=\frac{n}{(r+t)^n}\int_r^\rho\left\{\frac{n|c_0|x+|c_1|t^2}{\left(x^2+t^2\right)n|c_0|+2t^2|c_1|t}\right\}(x+t)^n\,dx\,. \quad (2.16)$$

For $0<r\le x\le\rho\le t$, by Lemma 2.5, we have

$$\frac{\left(n|c_0|x+|c_1|t^2\right)(x+t)}{\left(x^2+t^2\right)n|c_0|+2x|c_1|t^2}\le\frac{\left(n|c_0|\rho+|c_1|t^2\right)(\rho+t)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}.$$

(2.17)

Again using (2.17) in (2.16), we get

$$I\le\frac{n(\rho+t)}{(r+t)^n}\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\int_r^\rho(x+t)^{n-1}dx$$

$$=(\rho+t)\frac{\left(n|c_0|\rho+|c_1|t^2\right)}{\left(\rho^2+t^2\right)n|c_0|+2\rho|c_1|t^2}\left\{\left(\frac{\rho+t}{r+t}\right)^n-1\right\}.$$

(2.18)

Again, from the value of the integral on the right hand side of inequality (2.11) in the proof of Lemma 2.8, the value of the integral (2.15) is $\left[\left\{\frac{\left(\rho^2+t^2\right)n|c_0|+2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0|+2t^2|c_1|r}\right\}^{\frac{n}{2}}-1\right]$, and the conclusion of the lemma immediately follows from inequality (2.18).

**Lemma 2.11.** If $f(z)$ is a polynomial of degree $n$ and $g(z)=z^n\overline{f\left(\frac{1}{\bar z}\right)}$, then for each $\alpha$, $0\le\alpha<2\pi$ and $r>0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|g'\left(e^{i\theta}\right)+e^{i\alpha}f'\left(e^{i\theta}\right)\right|^r d\theta\,d\alpha\le 2\pi n^r\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^r d\theta\,.$$

(2.19)

The above lemma is due to Aziz and Rather [2].

**Lemma 2.12.** Let z be complex and independent of $\alpha$, where $\alpha$ is real, then for $p>0$,

$$\int_0^{2\pi}\left|1+ze^{i\alpha}\right|^p d\alpha=\int_0^{2\pi}\left|e^{i\alpha}+|z|\right|^p d\alpha\,. \quad (2.20)$$

This lemma belongs to Gardner and Govil [6].

## 3. PROOF OF THE THEOREM

Since the polynomial $f(z)$ has no zero in $|z|<t$, $t>0$, the polynomial $F(z)=f(\rho z)$ has no zero in $|z|<\frac{t}{\rho}$, $\frac{t}{\rho}\ge 1$. By applying Lemma 2.2 to $F(z)$, we have for $|z|=1$,

$$\frac{t}{\rho}\left|F'(z)\right| \le \left|G'(z)\right| \text{ for } |z| = 1, \qquad (3.1)$$

where $G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)}$.

We can easily verify that for every real number $\alpha$ and $R \ge r' \ge 1$,

$$\left|R + e^{i\alpha}\right| \ge \left|r' + e^{i\alpha}\right|.$$

This implies for each $q > 0$,

$$\int_0^{2\pi} \left|R + e^{i\alpha}\right|^q d\alpha \ge \int_0^{2\pi} \left|r' + e^{i\alpha}\right|^q d\alpha. \qquad (3.2)$$

For points $e^{i\theta}$, $0 \le \theta < 2\pi$, for which $P'\left(e^{i\theta}\right) \ne 0$, we denote

$$R = \left|\frac{Q'\left(e^{i\theta}\right)}{P'\left(e^{i\theta}\right)}\right|, \text{ and } r' = \frac{k}{\rho} \text{ then from (3.1),}$$

$$R \ge r' \ge 1.$$

Now, we have for each $q > 0$,

$$\int_0^{2\pi} \left|G'\left(e^{i\theta}\right) + e^{i\alpha}F'\left(e^{i\theta}\right)\right|^q d\alpha = \left|F'\left(e^{i\theta}\right)\right|^q \int_0^{2\pi} \left|\frac{G'\left(e^{i\theta}\right)}{F'\left(e^{i\theta}\right)} + e^{i\alpha}\right|^q d\alpha.$$

$$= \left|F'\left(e^{i\theta}\right)\right|^q \int_0^{2\pi} \left|\left|\frac{G'\left(e^{i\theta}\right)}{F'\left(e^{i\theta}\right)}\right| + e^{i\alpha}\right|^q d\alpha \text{ (by Lemma 2.12)}$$

$$\ge \left|F'\left(e^{i\theta}\right)\right|^q \int_0^{2\pi} \left|\frac{t}{\rho} + e^{i\alpha}\right|^q d\alpha, \text{ [by (3.2)]}. \qquad (3.3)$$

For points $e^{i\theta}$, $0 \le \theta < 2\pi$, for which $F'\left(e^{i\theta}\right) = 0$, inequality (3.3) trivially holds.

Now using (3.3) in Lemma 2.11, we obtain for each $q > 0$,

$$\int_0^{2\pi} \left|\frac{t}{\rho} + e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} \left|F'\left(e^{i\theta}\right)\right|^q d\theta \le 2\pi n^q \int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^q d\theta,$$

which is equivalent to

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|F'\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le nS_q \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}},$$

where

$$S_q = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{t}{\rho} + e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$$

Since $F(z) = f(\rho z)$, $F'(z) = \rho f'(\rho z)$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le \frac{n}{\rho} S_q \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}},$$

This in conjunction with Lemma 2.8 and noting $\frac{S_q}{\rho} = T_q$, we obtain

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f'\left(\rho e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}} \le nT_q \left\{\frac{1}{2\pi}\int_0^{2\pi} \left[\left|f\left(re^{i\theta}\right)\right| + M(f,r) \times \left\{\left(\frac{\left(\rho^2+t^2\right)n|c_0| + 2t^2|c_1|\rho}{\left(r^2+t^2\right)n|c_0| + 2t^2|c_1|r}\right)^{\frac{n}{2}} - 1\right\}\right]^q d\theta\right\}^{\frac{1}{q}}$$

This completes the proof of the Theorem.

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